LP model

$\text{Min } z_0 = \vec{C}\vec{X}$

s.t $\quad A\vec{x} = \vec{b}$; A is $m \times n$ matrix

All $\vec{x}_j \geq 0$, $j = 1, \ldots, n$.

$$A = [B, N], \quad \vec{x} = [\vec{x}_B, \vec{x}_N], \quad \vec{c} = [\vec{c}_B, \vec{c}_N]$$

$B\vec{x}_B = \vec{b}$, if B^{-1} exists

then, $\vec{x}_B = B^{-1}\vec{b}$

Notations:

column vectors

(i) $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$

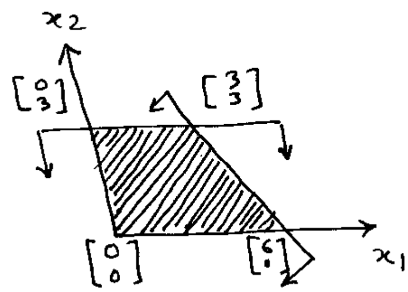
(ii) $\vec{y}_1 = B^{-1}a_1, \ \vec{y}_2 = B^{-1}a_2, \ldots, \ \vec{y}_n = B^{-1}\vec{a}_n$

(iii) $\vec{\omega} = \vec{c}_B B^{-1}$

Linear optimization

Example. Find basic solutions.

$$\boxed{\begin{aligned} x_1 + x_2 &\le 6 \\ x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}}$$



$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 6 \\ x_2 \quad + x_4 &= 3 \end{aligned}$$

$$x_1, x_2 \ge 0$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\begin{array}{cccc} x_1 & x_4 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right] [\vec{x}] = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$A \vec{x} = \vec{b}$$

$$A = [B, N]$$

B is invertible matrix (2×2)

N is non-invertible matrix

$\vec{x} = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$ to the equation $A\vec{x} = \vec{b}$

$\vec{x}_B = B^{-1} b$, $x_N = 0$

$\vec{x}_B \ge 0$, then x is basic feasible solution

B is called basic matrix

N is called nonbasic matrix

Components of x_B are called <u>basic variables</u>

components of x_N are called <u>nonbasic variables</u>

Finding basic solutions;

1) $$B = [\vec{a}_1, \vec{a}_4] = \begin{array}{c} \begin{array}{cc} x_1 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \quad \vec{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = B^{-1} \vec{b}$$

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \rightarrow \text{Basic feasible solution}$$

2) $$B = [\vec{a}_2, \vec{a}_4] = \begin{array}{c} \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \end{array}, \quad \vec{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = B^{-1} \vec{b}$$

$$\vec{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} \rightarrow \text{Basic solution.}$$

Hence, a multitude of basic solutions can be found by forming various combinations of column vectors in B matrix. Which B matrix combination is best? It depends upon objective function. Simplex method is a method to find optimum solution for LP problems.

Simplex Method

i) Start with initial basic solution with B as invertible matrix; then

$$\vec{x}_B = B^{-1}\vec{b}$$

ii) Solution $Z_0 = \vec{C}_B \vec{\bar{B}}$

iii) Optimality test; (Determine entering variable)

a) find $\vec{w} = \vec{C}_B B^{-1}$

b) for all non-basic variables; Calculate $z_j - c_j$

where, $z_j - c_j = \vec{w} a_j - c_j$

and let $z_k - c_k = \max\{z_j - c_j\}$

c) IF $z_k - c_k > 0$, then solution is not optimal, and variable x_k will enter basic solution.

d) IF $z_k - c_k \leq 0$, then OPTIMAL solution; STOP

iv) Find Leaving Variable;

For entering variable x_k, find $\vec{y}_k = B^{-1}\vec{a}_k$

a) IF all $\vec{y}_k \leq 0$, STOP; solution is unbounded.

b) Perform Minimum ratio test for all $y_{ik} > 0$;

Entering variable x_r will satisfy minimum ratio test as follows :

$$\frac{B^{-1}\vec{b_r}}{y_{rk}} = \underset{i \in R}{\mathrm{Min}} \left\{ \frac{B^{-1}\vec{b}}{y_{ik}} \right\} \; ; \; y_{ik} > 0$$

where i is the index of column vector $\vec{y_k}$ for which $y_{ik} > 0$.

Actually; x_r is the leaving variable from basic variable set $\vec{x_B}$ satisfying the following equation;

$$\vec{x_B} = B^{-1}\vec{b} - B^{-1}\vec{a_k} \cdot x_k$$

v. a) Update B matrix by replacing $\vec{a_r}$ with $\vec{a_k}$.

b) Update $\vec{c_B}$ (vector of basic variable coefficients)

Goto step (ii)

Example: Solve LP model using Simplex Method.

$$\text{Min.} \quad x_1 + x_2 - 4x_3$$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + 2x_3 &\le 9 \\ x_1 + x_2 - x_3 &\le 2 \\ -x_1 + x_2 + x_3 &\le 4 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

soln:

Introduce slack variables, x_4, x_5 and x_6

$$\text{Min} \quad x_1 + x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + 2x_3 + x_4 \qquad\qquad &= 9 \\ x_1 + x_2 - x_3 \qquad + x_5 \qquad &= 2 \\ -x_1 + x_2 + x_3 \qquad\qquad + x_6 &= 4 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

Initial basis $B = \begin{array}{c} \vec{a}_4 \ \vec{a}_5 \ \vec{a}_6 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$, Basic soln $\rightarrow \vec{x}_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$, $x_4 = 9$, $x_5 = 2$, $x_6 = 4$

obj. fn $\rightarrow z = \vec{c}_B \vec{X}_B = (0,0,0) \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix} = 0$

Determine Entering variable; $\quad \omega = c_B B^{-1} = (0,0,0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = (0,0,0)$

$$z_1 - c_1 = \omega a_j - c_1 = c_B B^{-1} a_1 - c_1 = 0 - (1) = -1$$

$$z_2 - c_2 = \omega a_2 - c_2 = c_B B^{-1} a_2 - c_2 = 0 - (1) = -1$$

$$(\leftarrow) \quad z_3 - c_3 = \omega a_3 - c_3 = c_B B^{-1} a_3 - c_3 = 0 - (-4) = 4$$

Since $z_3 - c_3 > 0$, x_3 enters the basis;

Calculate; $y_3 = B^{-1} a_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} +2 \\ -1 \\ +1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Determine leaving variable by Minimum ratio test;

$$\text{Minimum}\left\{ \frac{\bar{b}_1}{y_{13}}, \frac{\bar{b}_2}{y_{23}}, \frac{\bar{b}_3}{y_{33}} \right\} = \min\left\{ \frac{9}{2}, \frac{-2}{1}, \frac{4}{1} \right\}$$

Therefore, the index $r = 2$; x_6 leaves the solution

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \\ x_{B_3} \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} x_3$$

and x_6 first drops to zero when $x_3 = 4$

<u>Iteration 2</u>

Variable x_3 enters the basis and x_6 leaves.

$$B = \begin{array}{c} \begin{matrix} a_4 & a_5 & a_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \quad x_B = B^{-1} b = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

obj. fn: $z = c_B x_B = (0, 0, -4) \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = -16$

Calculate $\vec{\omega} = \vec{c}_B B^{-1} = (0, 0, -4) \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = (0, 0, -4)$

find

$$z_1 - c_1 = \omega a_1 - c_1 = (0, 0, -4) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - 1 = 3$$

$$z_2 - c_2 = \omega a_2 - c_2 = (0, 0, -4) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 = -5$$

since $z_1 - c_1 > 0$ and maximum; $\underline{x_1}$ enters basis.

$$y_1 = B^{-1} a_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ +1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ +1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

Minimum ratio test;

$$\text{Min} \left\{ \frac{\bar{b}_1}{y_{11}}, \frac{\bar{b}_2}{y_{21}}, \frac{\bar{b}_3}{y_{31}} \right\} = \text{Min} \left\{ \overset{x_4 \downarrow}{\frac{1}{3}}, \overset{x_5 \downarrow}{\frac{6}{0}}, \overset{x_3 \downarrow}{\frac{4}{-1}} \right\}$$

Hence, x_4 leaves the basis (why)

$$\begin{bmatrix} x_{B1} \\ x_{B2} \\ x_{B3} \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}}_{B^{-1}b} - \underbrace{\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}}_{B^{-1}a_1} x_1$$

x_4 becomes zero when $x_1 = 1/3$;

$\boxed{\text{Iteration 3}}$ $\underline{x_1}$ enters basis in place of $\underline{x_4}$

$$B = \begin{array}{c} \begin{matrix} a_1 & a_5 & a_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 \\ +1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \end{array}, \quad x_B = B^{-1} \cdot \vec{b} = \begin{bmatrix} 1 & 0 & 2 \\ +1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 6 \\ 13/3 \end{bmatrix}$$

$$Z = c_B \cdot x_B = (1, 0, -4) \begin{bmatrix} 1/3 \\ 6 \\ 13/3 \end{bmatrix} = -17$$

<u>Optimality test</u>

$$\omega = c_B B^{-1} = (1, 0, -4) \begin{bmatrix} 1 & 0 & 2 \\ +1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = (-1, 0, -2)$$

find $z_2 - c_2 = \omega a_2 - c_2 = (-1, 0, -2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 = -4$

In case of no-cycling, optimal solution <u>is at hand</u>.